Recent Advances in Dynamical Systems
GDM2024

8-12 July, 2024, Santiago de Compostela, Spain

Abstracts

Monday, 8th July

Recent Advances in Nonlinear Dynamics

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Exploring a tentacular geometry

Pablo S. Casas, Fátima Drubi and Santiago Ibáñez

We investigate a one-parameter family of time-reversible Hamiltonian vector fields in $\mathbb{R}^4$, motivated by its pivotal role in studying the generic unfoldings of the four-dimensional nilpotent singularity of codimension four. Beyond its theoretical importance, this system finds applications in domains like Physics and Engineering, driving extensive treatment in previous literature.
Within the system, a bifocal equilibrium point emerges for a range of parameter values. The corresponding two-dimensional invariant manifolds, both stable and unstable, intricately fold in the phase space, forming complex patterns known as "tentacular geometry" (see [1]). Our objective is to understand the evolution of this tentacular geometry with respect to the parameter. To achieve this, we analyze the traces left by invariant manifolds as they intersect a Poincaré section invariant under the reversibility map. Our methodology allows us to establish connections between the tentacular geometry on the cross-section and the study of cocoon bifurcations (see [3]).
Moreover, we contextualize these findings by exploring the significance of studying the unfolding of the four-dimensional nilpotent singularity of codimension four and the challenges it presents.
These results are collected in [2].

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Capturing the dynamics of a barred galaxy from a single snapshot

P. SÁNCHEZ-MARTÍN, J. AMORÓS AND J. MASDEMONT

The Gaia telescope's recent data release (Gaia DR3) provides highly accurate information regarding the positions and velocities of stars in the Milky Way and nearby regions, such as the Large Magellanic Cloud. This dataset requires methods for identifying primary features of the galaxy such as its central bar and arms from a single snapshot, and offers a unique opportunity to validate the explanation of these features through Invariant Manifold Theory. To overcome the issues of conventional methods (which rely on density regions), we propose a new one for identifying the main galactic features from a single snapshot, and validate it using galactic simulations that offer multiple snapshots of their evolution over time.

The Dynamical Systems approach to studying galactic shape and dynamics, referred to as Invariant Manifold Theory in Astrophysics and developed in previous works, consists in studying barred galaxies as dynamical systems with a gravitational potential generated by the main components of the galaxy, such as the bar, disc, bulge, and halo. Typically, this system has five equilibrium points. Among these points, the center and the triangular points are linearly stable elliptic points, while the collinear points at the ends of the bar are hyperbolic points. Surrounding these collinear points, there exist families of planar and vertical Lyapunov periodic orbits. The associated invariant manifolds (both stable and unstable) of the planar family outline the arms and rings of barred galaxies. Orbits within these manifolds enable the transfer of matter between the inner and outer regions of the galaxy, with the Lyapunov orbits acting as gates.

Our goal is to identify from a single snapshot the central bar of the galaxy and determine its position, size, pattern speed, and density distribution, as well as to distinguish the galactic arms. These characteristics will be used to model the potentials of the various galaxy components. When combined with the pattern speed, they will be incorporated into the classical dynamical system model. In galactic simulations, we will then compare the dynamics from the equations of motion governing this galactic model with the evolution of the galaxy in the simulation to enhance our understanding of arm formation.

Once validated through simulations, our methodology can be applied to data from the Large Magellanic Cloud, a barred galaxy known for its asymmetrical density distribution in its central region and among its arms.

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Deepening the rocking motion dynamics

S. Pérez-González, J.S. Pérez del Río and F. Prieto

In a pioneering work, Housner [1] established the dynamics of a rigid block on an equally rigid surface, under the influence of an external impulse. Despite the apparent simplicity of this problem, the phenomenon of Rocking Motion (RM) has proven to be highly complex in certain circumstances. The complexity of this behaviour arises from two main factors: 1) the discontinuous nature of its dynamics, where the solid abruptly transitions from a stationary state to oscillating around different rotation points, and 2) the intermittency of the impact of the block with the underlying surface. It is precisely this complexity that keeps the study of the dynamics of Rocking Motion an unsolved problem to this day [2].
On the other hand, in [3], a formulation was presented that managed to convert the segmented equations of classical Rocking Motion into a set of continuous differential equations. In addition, the structure of the impact forces, which in Housner's model were represented by a restitution coefficient, was now interpreted as a simple coupling that explained the existence of instantaneous impulsive terms, known as Dirac forces. It is relevant to highlight that the regularization employed in [3] allows for addressing broader dynamical systems.
Starting from the system of central forces that describes Rocking Motion, where angular momentum is a conserved quantity, in our work we managed to generalize it into a singular Liènard-type system that admits a certain regularization. This allows to analyse the complete dynamics of the system and its corresponding 2-parameter bifurcation diagram.

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Tuesday, 9th July

Spiral waves, L_3 and beyond all order phenomena

I. BALDOMÁ, M. AGUARELES, M. GIRALT, M. GUARDIA, T. M-SEARA

We present two different settings where a beyond all order phenomenon occurs by means of exponentially small quantities with respect to a suitable singular parameter. It summarizes different joint works with Maria Aguareles, Mar Giralt, Marcel Guardia and Tere M-Seara.
In the first instance, we consider the restricted planar circular 3-body problem (RPC3BP), namely the planar motion of a massless body under the gravitational influence of two massive bodies, called the primaries, which are assumed to move in a circular motion. By setting a rotating reference system, one has that the primaries are fixed equilibrium points. In addition, the system also possesses the well known Lagrangian-Euler equilibrium points. We focus on the L_3 Lagrangian point which in our regime is of saddle-center type.
Exploiting the fast oscillations with respect to μ, the ratio between the masses of the primaries parameter, we have measured the distance between the one dimensional stable and unstable manifold of L_3 when the mass ratio goes to 0. This distance turns out to be exponentially small in μ and therefore classical perturbative methods (like the Melnikov-Poincaré method) do not apply. The dynamical implications of the result are a) existence of homoclinic connections for some values of μ, b) the existence of chaotic motions which pass exponentially close to L_3 and c) Newhouse phenomena. This is a joint work with M. Giralt and M. Guardia.
The second example is radically different: existence of spiral patterns in reaction-diffusion systems. Spiral patterns are commonly observed in certain chemical, biological and physical systems as governed by a chemical or biological reaction and a spatial diffusion: the so called reaction-diffusion systems.
We focus on the existence of rotating archimedian spirals for the Ginzburg-Landau systems which corresponds to the first order of a reaction-diffusion equation near a Hopf bifurcation. Such spirals are solutions of a three dimensional system of ordinary differential equations depending on two parameters, the twist parameter, q and the asymptotic wavenumber, k. Moreover, the spiral solutions has to satisfy four boundary conditions which forces to a selection mechanism for k with respect to q. We have proven that, in order to exist rotating archimedian spirals for the Ginzburg-Landau systems, the asymptotic wavenumber has to be $k = k(q) \sim \frac{1}{q} A_n e^{-\frac{\pi}{2nq}}$ with A_n a constant that only depends on the solutions for $q = 0$.
Many authors, Koppel, Hagan, Greenberg, Chapman, etc. have previously studied this problem using different techniques (Fenichel's theory, asymptotic methods, numerical methods, shooting methods) and conjecturing the formula we finally prove. Our study is based on functional and complex analysis techniques. This is a joint work with M. Aguareles (U. de Girona) and T.M. Seara (U. Politècnica de Catalunya).

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Periodic orbits near collision in a restricted four-body problem for the eight choreography

Abimael Bengochea, Jaime Burgos-García and Ernesto Pérez-Chavela

We study doubly symmetric periodic orbits near collision in a non-autonomous restricted planar four-body problem. The restricted problem consists of a massless particle moving under the gravitational influence due to three bodies following the figure-eight choreography. We use regularized coordinates in order to deal numerically with motions near collision, and reversing symmetries to characterize the symmetric periodic near collision. The symmetric periodic orbits (initial conditions) were computed by means of solving numerically the solution of some boundary-value problems. We explain theoretically, and confirm numerically, how different parts of the diagram of initial conditions are related due to the symmetry of the figure-eight choreography.

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Chaos and thermal equilibrium in a chain of classical dipoles

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The relationship between chaos, thermalization and ergodicity in dynamical systems with a large number of degrees of freedom is an old topic that began with the pioneering study of Fermi, Pasta, Ulam and Tsingou (FPUT) [1] in 1953, and that remains an open question nowadays. In this talk we address the connection between chaos, thermalization and ergodicity in a linear chain of N interacting dipoles. Starting from the ground state, and considering chains of different numbers of dipoles, the system is excited with a given excess energy ΔK. In order to follow the time evolution of the chaoticity of the system and the energy localization along the chain, we computed, up to a very long time, the statistical average of the finite time Lyapunov exponent $\lambda(t)$ [2] and the participation ratio $\Pi(t)$ [3], respectively. For small excess energy ΔK, the evolution of $\lambda(t)$ and $\Pi(t)$ indicates that the system becomes chaotic at approximately the same time as $\Pi(t)$ reaches a steady state. For the largest considered values of ΔK the system becomes chaotic at an extremely early stage in comparison with the energy relaxation times. We find that this fact is due to the presence of chaotic breathers [4] that keep the system far from thermalization and ergodicity. Finally, we show numerically and analytically that the asymptotic values attained by the participation ratio $\Pi(t)$ fairly correspond to thermal equilibrium. More complete information can be found in [5].

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Splitting of separatrices in generalized standard maps

Dídac Gil Rams, Inmaculada Baldomà Barraca and Pau Martín de la Torre

We study transversal intersections between the invariant manifolds (stable and unstable) associated to an hyperbolic fixed point for a class of maps. These intersections are known as homoclinic orbits. The existence of these kind of orbits is one of the most celebrated methods to prove the existence of chaotic dynamics in a system. Indeed the Morse-Smale theorem ensures that if there exist transversal intersections between the invariant manifolds of the same invariant object, the system is locally conjugate to a Smale horseshoe with infinite symbols.
The classical Melnikov theory is a first order perturbative theory that, in addition, can be used to measure the intersection angle between the invariant manifolds. However straightforwardly there are cases where the Melnikov function is exponentially small and the associated theory is not true. In these cases, to measure, for example, the intersection angle between the manifolds, becomes a difficult and technical task, since it is a beyond all orders phenomenon. This is the case of the problem we are considering.
We study the splitting of separatrices on generalized standard maps firstly defined in [2]. This generalization includes the already studied maps like *the standard map*, first studied by Lazutkin [1], or *the perturbed McMillan map*.
More concretely, we are going to study the intersection of the invariant manifolds associated to a fixed point of the discrete dynamical system

$$\begin{cases} x^* = x + y + f(x,h), \\ y^* = y + f(x,h), \end{cases}$$

where h is a small parameter and f depends analytically on $|h| < h_0$, $|x| < \rho_0$, for some fixed $h_0, \rho_0 > 0$. We consider f to be of the form

$$f(x,h) = \sum_{k\geq 0} f_k(x) h^{k+2} \quad \text{where} \quad f_k(x) = \begin{cases} \sum_{j=1}^{d_k} f_{k,j} x^j & \text{named } \textit{polynomial} \text{ case,} \\ \sum_{j=-d_k}^{d_k} f_{k,j} e^{ijx} & \text{named } \textit{trigonometric} \text{ case,} \end{cases}$$

and $f_{k,d_k} \neq 0$. In addition some extra condition on the exponents d_k are imposed.
We obtain an asymptotic formula for the *Lazutkin invariant*, value related to the area between two homoclinic points, and its first term depends on a *Stokes constant* that is generically different from zero. To do so, one of the techniques that we use is the *inner equation* related to our generalized standard maps. We also study this Stokes constant using a computer assisted proof in order to obtain an interval for its value.

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Characterising blenders via covering relations and cone conditions

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A blender in dynamical systems is an invariant hyperbolic set, whose stable (unstable) manifold behaves as if it was of higher dimension than the dimension of the stable (unstable) bundle. The simplest example of a blender is an invariant hyperbolic set in $\mathbb{R}^3$ with a 1-dimensional stable manifold and a 2-dimensional unstable manifold, whose stable manifold behaves as a two dimensional plane. In the talk we will present a simple general construction, which leads to the existence of blenders. We will also show how blenders can be linked together to produce the so called robust 'heterodimensional cycles'. The method is fully constructive; we will present an example of application for a Hénon-like map in $\mathbb{R}^3$.

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Rigid systems in the plane. Overview and new results

M.J. Álvarez, J.L. Bravo and L.A. Calderón

In this presentation, we provide an overview of the current state of knowledge concerning rigid systems in the plane. Additionally, we present new results for a family of this class of systems.
Rigid system are characterized by having the origin as its unique critical point (which is always monodromic) and by all orbits around it having constant angular velocity. They can be written as

$$\begin{cases} \dot{x} = -y + xF(x,y), \\ \dot{y} = \quad x + yF(x,y). \end{cases}$$

There are several factors that make the rigid family interesting. Firstly, the fact that the origin is its only critical point implies that any potential limit cycles, if they exist, have to be nested around it. Secondly, this family plays an important role in the broader problem of isochronicity.
Our presentation focuses on the quartic rigid family, that is the simplest non-trivial polynomial family of rigid systems without rotatory parameters. We study this family within the finite plane and on the Poincaré sphere. Within this family we prove that for a significant subfamily there are no limit cycles in the plane. However, when the critical points at infinity are cusps (the generic case), a periodic orbit crossing the equator of the sphere always exists, when no homoclinic nor heteroclinic orbits exist.

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Invariants and reversibility in polynomial systems of ODEs

VALERY ROMANOVSKI, MATEJA GRAŠIČ, ABDUL SALAM JARRAH

I present the results of the paper [1] which explores a relationship between invariants of certain group actions and the time-reversibility of two-dimensional polynomial differential systems exhibiting a $1 : -1$ resonant singularity at the origin. We focus on the connection of time-reversibility with the Sibirsky subvariety of the center (integrability) variety [2], which encompasses systems possessing a local analytic first integral near the origin. An algorithm for generating the Sibirsky ideal for these systems is proposed and the algebraic properties of the ideal are examined.

Furthermore, using a generalization of the concept of time-reversibility [4, 3] we study n-dimensional systems with a $1 : \zeta : \zeta^2 : \cdots : \zeta^{n-1}$ resonant singularity at the origin, where n is prime and ζ is a primitive n-th root of unity. We study the invariants of a Lie group action on the parameter space of the system, leveraging the theory of binomial ideals as a fundamental tool for the analysis. Our study reveals intriguing connections between generalized reversibility, invariants, and binomial ideals, shedding light on their complex interrelations.

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Wednesday, 10th July

"Large" strange attractors associated to heteroclinic bifurcations

ALEXANDRE RODRIGUES

In this talk, I will present a mechanism for the emergence of strange attractors in a one-parameter family of differential equations defined in a three-dimensional sphere. When the parameter is zero, the flow exhibits an asymptotically stable heteroclinic network (*Bykov* network) formed by two one-dimensional connections and a two-dimensional separatrix between two hyperbolic saddle-foci with different Morse indices.
By moving the parameter, keeping the one-dimensional connections unchanged, I will focus on the case in which the two-dimensional invariant manifolds of equilibria do not intersect. I will show that, for a set of parameters sufficiently close to zero with positive Lebesgue measure, the dynamics exhibits "large" strange attractors winding around the "ghost" of a two-dimensional torus in phase space.
I also describe the existence of a sequence of parameter values for which the family exhibits a (super) stable sink. I emphasise the bifurcations that occur in the transition from an (asymptotically stable) network to a "large" strange attractor.
I relate new results with the existing literature. The contents of this talk may be found in [1, 3, 2, 4].

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In search of 2-D strange attractors for the 2-D quadratic family

A. MARQUÉS-LOBEIRAS, A. PUMARIÑO, J. Á. RODRÍGUEZ, AND E. VIGIL

Homoclinic tangencies for generic families of dissipative 3-D diffeomorphisms were classified by Tatjer [4]. For a certain type of generalized tangency involving a two-dimensional unstable manifold, the return maps have as their limit family the two-dimensional quadratic family

$$T_{a,b}(x,y) = (a + y^2, x + by) \tag{1}$$

As the study of the classical quadratic family was seminal for the proof of the persistence of 1-D strange attractors, the study of family (1) should be a first step towards the proof of the persistence of 2-D strange attractors.
A first analytical approach to the study of the dynamics of family (1) was made by searching for parameter values for which there exist invariant domains [2]. Attractors contained in such invariant domains were then numerically simulated [3]. Furthermore, a numerical analysis based on the Lyapunov exponents allowed to classify the attractors numerically detected for a certain region of parameter values into four types: periodic orbits, invariant closed curves, 1-D strange attractors, and 2-D strange attractors. See Figure 1 [3].
In order to analytically prove the existence and persistence of the 2-D strange attractors numerically found for family (1) and explain the bifurcations between the different types of attractors, we first carry out a study on the periodic orbits of low period and construct invariant domains with non-empty interior for sharp regions of parameter values. Also, we numerically simulate attractors contained in such domains and find similarities between them and the attractors of the 2-D tent maps family [1].

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Full Lyapunov exponents spectrum with deep learning from single-variable time series

Carmen Mayora-Cebollero, Ana Mayora-Cebollero, Álvaro Lozano and Roberto Barrio

Lyapunov Exponents are a classical tool used to study the behaviour of a dynamical system. Standard algorithms that approximate the full Lyapunov Exponents spectrum use the whole variables of the system [1] and, if only one variable is used, just the first Lyapunov Exponent (maximum Lyapunov Exponent) is computed [1, 2]. Deep Learning techniques have shown good performance in chaotic analysis (even a dense 3D analysis has been obtained for the Lorenz system) [3]. Therefore, we propose to use Deep Learning to approximate Lyapunov Exponents. In this presentation [4], we show how Deep Learning neural networks can be used to obtain a good approximation of the full Lyapunov Exponents spectrum of a dynamical system with only short single-variable time series. This strategy allows to speed up the complete analysis of a dynamical system using partial data.

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Reading multiplicity in unfoldings from ε-neighborhoods of orbits

VESNA ŽUPANOVIĆ, RENATO HUZAK, PAVAO MARDEŠIĆ, MAJA RESMAN

In [2] we study the relationship between the multiplicity of a fixed point of a function g, and the dependence on ε of the length of ε-neighborhood of any discrete orbit of g, tending to the fixed point. We study the space of functions having a development in a Chebyshev scale and use multiplicity with respect to this space of functions. We introduce critical Minkowski order, the notion recovering the relationship between the multiplicity of fixed points and the dependence on ε of the length of ε-neighborhood of orbits. In [1] we introduce a parameter in the generator function g. Using the new notion of continuous ε-neighborhood of a discrete orbit we study saddle-node, transcritical and pitchfork bifurcation. We obtain the asymptotic expansion of the length of the continuous ε-neighborhood of a discrete orbit.

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Singularly perturbed systems of ODEs with dependence on the fast time

IACOPO P. LONGO, RAFAEL OBAYA, ANA M. SANZ

Coupled slow and fast motions generated by ODEs are considered, with dependence on the fast time:

$$\begin{cases} \dot{x} = f(x,y)\,, \\ \varepsilon\,\dot{y} = g\left(x, y, \dfrac{t}{\varepsilon}\right), \end{cases}$$

where $\varepsilon > 0$ is a small parameter. By using the theory of nonautonomous dynamical systems for the fast variables, we obtain a new dynamical interpretation of the limit behaviour of the fast motion as the small parameter tends to zero. Besides, the results can be interpreted as a process of tracking of nonautonomous attractors. Some simulations are presented to illustrate the results.

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A dynamical analysis of Coupled Brusselator System

Ana Mayora-Cebollero, Roberto Barrio, Fátima Drubi, Santiago Ibáñez, Jorge A. Jover-Galtier, Álvaro Lozano, Carmen Mayora-Cebollero and Lucía Pérez

Brusselator model is a theoretical model that represents an autocatalytic chemical reaction with oscillations as the well-known Belousov-Zhabotinsky reaction. Coupling by diffusion two identical Brusselators, the Coupled Brusselator System (2-CBS) is obtained. This model presents different dynamical regimes among which it is worth highlighting two chaotic regions in different zones of the parametric space ([1] and [2]). In this presentation, we add a new parameter to the 2-CBS model to conclude with continuation that these two chaotic regions are not connected and consequently, a new small chaotic region is found [3]. Besides, we perform a detailed study of the synchronization phenomena [4] of the 2-CBS and 3-CBS (model obtained coupling three identical Brusselators).

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Fractal codimension in two-dimensional slow-fast systems

ANSFRIED JANSSENS, RENATO HUZAK AND GORAN RADUNVIĆ

This is a joint work with Prof Renato Huzak (University of Hasselt, Hasselt, Belgium), Peter De Maesschalck (University of Hasselt, Hasselt, Belgium) and Goran Radunvić. (University of Zagreb, Zagreb, Croatia).

My talk will be base on the papers [1] and [2] and some new results for the unsolved problems of these papers. In this presentation we introduce the notion of fractal codimension of a nilpotent contact point p, for $\lambda = \lambda_0$, in smooth planar slow–fast systems $X_{\epsilon,\lambda}$ when the contact order $n_{\lambda_0}(p)$ of p is even, the singularity order $s_{\lambda_0}(p)$ of p is odd and p has finite slow divergence, i.e., $s_{\lambda_0}(p) \leq 2(n_{\lambda_0}(p) - 1)$. The fractal codimension of p is a generalization of the "traditional" codimension of a slow-fast Hopf point of Liénard type, introduced in [3] and it is intrinsically defined, i.e., it can be directly computed without the need to first bring the system into its normal form. The intrinsic nature of the notion of fractal codimension stems from the Minkowski dimension of fractal sequences of points, defined near p using the so-called entry-exit relation, and slow divergence integral. We apply our method to a slow-fast Hopf point and read its degeneracy (i.e., the first nonzero Lyapunov quantity) as well as the number of limit cycles near such a Hopf point directly from its fractal codimension. We demonstrate our results numerically on some interesting examples by using a simple formula for computation of the fractal codimension.

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Chaos identification using Lagrangian-like descriptors

PÉREZ-PALAU, DANIEL, PICO LACHE, DIEGO AND ARRANZ-SANZ, SANTIAGO

Dynamic indicators emerged as a tool to comprehend different behaviours of dynamical systems. In this talk we will review the use and definitions of Finite-Time Lyapunov Exponents[2] and Lagrangian Descriptors[1]. Using the ideas introduced by those objects we will present the Discrete Lagrangian Descriptors as an alternative version of the Lagrangian Descriptors. We will show how these versions can be used to identify regions close to periodic orbits where tori appear. They also highlight chaotic regions of the system.
We will compare the different descriptors with classical tools in dynamical systems like Poincaré sections and invariant manifolds in two systems: the Atwood's pendulum and an oscillator in a Schwarzschild black hole.

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Thursday, 11th July

Michaelis-Menten: A harmless but problematic planar system

Sebastian Walcher

The Michaelis-Menten reaction network is fundamental in the mathematical modelling and analysis of enzyme kinetics in biochemistry. The qualitative features of the corresponding (parameter-dependent) two dimensional differential equation are not hard to analyze.
Problems arise, however, when it comes to parameter identification from laboratory measurements. For historic reasons, and also due to possible slow-fast phenomena, various reductions to dimension one were introduced. It is known that there are singular perturbation reductions in some parameter regions, but rigorous and sharp quantitative estimates for the approximation error were determined only a short time ago. In addition, recent work sheds light on a scenario for which no effective reduction to dimension one exists.
The mathematical tools include singular perturbation reduction (in a coordinate-independent version), a little algorithmic algebra (to identify critical parameter values from which singular perturbations emanate), and differential inequalities for cooperative systems.
In the course of the talk, the general approach to modelling chemical mass action reaction networks will be sketched, and some properties of the corresponding ordinary differential equations will be mentioned.

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Limit cycles for real and complex planar fewnomial vector fields

ARMENGOL GASULL

In Hilbert's 16th problem the main goal is to get (good) upper and lower bounds, for the number of limit cycles of a planar polynomial differential system in terms of its degree. In this talk we face the same question but in terms of the number of monomials of the planar polynomial differential equation. This number of monomials can be counted either in the usual real notation or in the complex notation, $\dot{z} = F(z, \bar{z})$, and both points of view are, not at all, equivalent.

More precisely, in the real notation, let $\mathcal{M}_m$ be the family of planar polynomial vector fields with at most m monomials. We define the *Hilbert monomial number*,

$$\mathcal{H}^M(m) = \sup\{\text{number of limit cycles of } X : \ X \in \mathcal{M}_m\}.$$

So far very little is known about $\mathcal{H}^M(m)$. It follows from the recent work [3] that $\mathcal{H}^M(m) = 0$ for $m \in \{1, 2, 3\}$, $\mathcal{H}^M(m) \geqslant m - 3$ for $m \geqslant 4$ and that there is a sequence $(m_k) \subset \mathbb{N}$, with $m_k \to \infty$, such that $\mathcal{H}^M(m_k) \geqslant N(m_k)$, with $N(m)$ of order $O(m \ln m)$. Our main result on this problem is an improvement of these general lower bounds proving that $\mathcal{H}^M(m)$ increases at least with order $O(m^2)$, see [5].

Concerning the complex notation, on one hand in [4] it is proved that for complex differential equations with three monomials, $\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q$, with k, l, m, n, p, q non-negative integers, there is no upper bound for the number of limit cycles. On the other hand, for complex differential equations with two monomials this upper bound is known to be one, see [1].

In fact, if $N = \max(k + l, m + n, p + q)$ and $H_3(N) \in \mathbb{N} \cup \{\infty\}$ denotes the maximum number of limit cycles of the above systems with this restriction, it is proved in [4] that $H_3(N) \geq (N + 3)/2$, for $N \geq 3$ odd. The second aim of this talk is to show our improvements of this lower bound and to present our results for trying to determine $H_3(2)$, see [2].

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The cyclicity of hyperbolic hemicycles

D. MARIN AND J. VILLADELPRAT

In this talk I will explain some recent results about the cyclicity of a type of unbounded polycycle Γ called hemicycle. Compactified to the Poincaré disc, Γ consists of an affine straight line together with half of the line at infinity and has two singular points, which are hyperbolic saddles located at infinity. The stability of this kind of polycycle was studied previously by Gasull, Mañosa and Mañosas in [4]. We consider quadratic integrable systems belonging to the class Q_3^R and having two hemicycles, Γ_u and Γ_ℓ, surrounding each one a center. We study the cyclicity of Γ_u and Γ_ℓ when perturbed inside the whole family of quadratic systems and the result that we obtain can be viewed as a contribution to the completion of the program of Dumortier, Roussarie and Rousseau to prove that $H(2)$ is finite (see [2]). We also study the number of limit cycles bifurcating simultaneously from Γ_u and Γ_ℓ when perturbed inside the whole family of quadratic systems. Finally we show that for three specific cases there exists a simultaneous alien limit cycle bifurcation from Γ_u and Γ_ℓ (see [1, 3]).

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Recent Advances in Nonlinear Dynamics
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Biological pattern formation beyond Turing: An attempt on characterizing all patterning gene regulatory networks

KEVIN MARTÍNEZ-ANHOM AND ISAAC SALAZAR-CIUDAD

Cell differentiation is the process by which complex patterns of gene expression arise and evolve in a developing embryo guiding the formation of its final complex anatomy. These pattern transformations can be mainly explained as a response to the interaction of biochemical signals sent from one cell to the other according to a gene regulatory network, that is, the network of activatory and inhibitory interactions between the different gene products involved in the signalling mechanism, [2].

In a seminal paper of 1952, [4], Alan Turing proved that a simple two-node network where the gene products diffuse and antagonistically interact with each other is enough to produce heterogeneous pattern solutions along a spatial domain. However, the fine tuning of model parameters that his theory predicts together with the apparent lack of empirical evidence have raised doubts on the feasibility of Turing's proposal as a mechanism for cell differentiation in real-life systems ever since. A simple question comes thus to mind, [3]: 'which gene regulatory networks can actually lead to proper pattern transformations?'.

Our theoretical analysis and numerical simulations, performed in the framework of the reaction-diffusion equations, $\partial_t \boldsymbol{u}(t,\boldsymbol{x}) = f(\boldsymbol{u}) + D\nabla^2 \boldsymbol{u}(t,\boldsymbol{x})$, show that, regardless of the immense number of patterning networks that one can think of (specially when the number of gene products in hand is large), they can all be sorted out into just three fundamental types of gene regulatory networks: T-networks, based on Turing's mechanism, [1]; H-networks, including the classical hierarchical gradient-based models, [5]; and a third brand-new type that we named OT-networks. Additionally, we also show how the dynamics of the underlying chemical ODE, $\partial_t \boldsymbol{u}(t) = f(\boldsymbol{u})$, can actually affect the structural and dynamical properties of the final arising pattern, specially in the case of OT-networks.

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Insights from piecewise linear and threshold modelling in neuroscience: networks and delays

STEPHEN COOMBES

To gain insight into the behaviour of complex biological neural networks it can be very informative to consider models built from piecewise linear and possibly discontinuous dynamical systems [1]. While such models offer a beautifully simplistic perspective, their inherent lack of smoothness poses challenges, limiting the application of conventional tools from smooth dynamical systems. Careful consideration is needed to establish conditions for the existence, uniqueness, and stability of solutions. In this presentation I will describe a variety of nonsmooth neural oscillators, and show how to analyse periodic orbits. Building on the approach for analysing a single node, I will show how to treat synchronous network states using a phase-amplitude reduction and the master stability function and present some network examples built from integrate-and-fire single cell spiking neurons and Amari type oscillators describing population activity [2]. Introducing delays between nodes, particularly in Amari style networks, aligns with the modelling of large-scale brain activity observed in EEG/MEG recordings. I will present some new mathematical tools that merge the dynamics of nonsmooth and delayed systems to address space-dependent axonal delays and their impact on spatio-temporal patterns of brain activity [3]. Finally, I will discuss outstanding challenges for when the delays are state-dependent and present preliminary results for a new form of biologically motivated white matter plasticity rule.

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Analysis of coupled FitzHugh-Nagumo systems: insights into chaotic dynamics

F. Drubi, S. Ibáñez, D. Noriega

While the interaction between systems exhibiting Hopf bifurcations has been extensively explored in the literature, many aspects remain unknown. In this study, we focus on a model comprising two linearly coupled FitzHugh-Nagumo systems, well-known simplifications of neuron models that exhibit Hopf bifurcations. Our investigation reveals a variety of interesting phenomena, including Hopf-Pitchfork and Hopf-Hopf singularities leading to remarkably rich bifurcation diagrams, incorporating features such as the bifurcation of invariant tori.

In this talk, we will dive into our findings regarding the chaotic behaviors that emerge in these scenarios. Notably, we uncover bifurcations that lead to jumpings in the size of chaotic attractors -— a phenomenon closely related to the concept of interior crisis introduced in [1]. By shedding light on these intricate dynamics, our study contributes to a deeper understanding of the complex interplay between coupled systems.

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Ejection-collision solutions from KAM tori in restricted N-body problems

Aitor Ayape, Jesús F. Palacián, and Patricia Yanguas

The main idea is to study ejection-collision solutions in several cases of the N-body problem, starting with the restricted three-body problem in the planar case. In these solutions the particle of negligible mass passes several times near one of the primaries before colliding with it. In the so-called lunar regime, the Hamiltonian is a perturbation of the Kepler problem. Due to the high-degeneracy of the equations of motion we need to apply a special KAM theorem to deal with the multi-scaled Hamiltonian system. Specifically we apply Han-Li-Yi theorem to prove the persistence of KAM tori filled in with quasi-periodic motions of the small particle. This is achieved by applying a combination of regularisation, normalisations and symplectic reductions. Second, from the rectilinear quasi-periodic motions we identify those corresponding to the ejection-collision solutions.
The analysis of these solutions in the planar case of the restricted three-body problem has been performed since the 1980s from both an analytical and a numerical point of view. Our approach is generalised to the planar restricted N-body problem where the small particle orbits around one of the bodies forming a central configuration and to the spatial case of the restricted three-body problem, concluding with the existence of these solutions in all these cases. The latter has been studied mainly numerically, hence the interest in applying the techniques mentioned for the search for analytical results.

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Friday, 12th July

On the Lambert problem with drag

ANTONIO J. UREÑA

The Lambert problem consists in connecting two given points in a given lapse of time under the gravitational influence of a fixed center. While this problem is very classical, we are concerned here with situations where friction forces act alongside the Newtonian attraction. Under some boundedness assumptions on the friction, there exists exactly one rectilinear solution if the two points lie on the same ray, and at least two solutions traveling in opposite directions otherwise.

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Constant sign derivatives of Green's function via spectral theory

Mouhcine Yousfi, Alberto Cabada and Lucía López-Somoza

Abstract

In this talk we will characterize the set of real parameters M in which certain partial derivatives of the Green's function g_M related to a linear operator

$$T_n[M]\,u(t) := u^{(n)}(t) + M\,u(t), \quad t \in I := [a,b],$$

coupled to different two-point conditions, are of constant sign on the square of definition $I \times I$. We will do this characterization without using the explicit expression of Green's function g_M.
The constant sign interval is characterized by the first eigenvalues of the operator $T_n[0]$, related to suitable boundary conditions.

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On the global dynamics of some Lotka-Volterra and Kolmogorov systems

Érika Diz-Pita, Jaume Llibre and M. Victoria Otero-Espinar

Lotka-Volterra and Kolmogorov differential systems can be used to model many natural phenomena, and therefore, from a mathematical point of view, it is of considerable interest to advance in the detailed understanding of their dynamic behavior.
The study of Lotka-Volterra systems on dimension two has been completed by D. Schlomiuk and N. Vulpe in [5], but for Lotka-Volterra systems on dimension three, there were only some partial results for special cases in which the systems were simpler.
We present here the study of a large subfamily of Lotka-Volterra systems in dimension three, those that have a first integral of the form $x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}$. Using Darboux's integrability theory, we have reduced the study of these systems to the study of two Kolmorogov systems in the plane, whose global dynamics in the Poincaré disk has been fully studied in [1, 2, 3, 4].
We will detail how different techniques and tools have been used to achieve the classification of the global dynamics of these systems, among them, Poincaré compactification, desingularization of equilibrium points by means of directional blow-up's, index theory or the use of topological invariants for the final topological classification.

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Recurrence relations with reflection

F. Adrián F. Tojo

In this talk we present an algebraic theory of linear recurrence equations and systems with constant coefficients and reflection. The goal of the theory is to reduce this kind of equations to ordinary recurrence relations, for which explicit solutions and qualitative behavior can be studied or is already known. We obtain explicit solutions and the Green's functions associated to different problems under general linear boundary conditions. Furthermore, we establish the relations between different structures of recurrence and differential operators, showing the similarities and differences between them.

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Oscillations for the forced pendulum equation under relativistic effects

Pablo Amster and José Ángel Cid

In this talk we will deal with the forced pendulum equation subject to relativistic effects; from the mathematical point of view we will obtain a φ-laplacian equation that presents some differences with respect to previously studied relativistic equations [1, 3, 4, 5]. Our main goal will be to discuss the existence of periodic solutions and present both the discrepancies and similarities with respect to previous results in the literature. This is a joint work with professor Pablo Amster from the University of Buenos Aires and it is based on [2] .

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